,

ON SOME NONCOMMUTATIVE SYMMETRIC FUNCTIONS ANALOGOUS TO HALL-LITTLEWOOD AND MACDONALD POLYNOMIALS

JEAN-CHRISTOPHE NOVELLI, LENNY TEVLIN, AND JEAN-YVES THIBON

To Christophe Reutenauer, on the occasion of his 60th birthday

Abstract. We investigate the connections between various noncommutative analogues of Hall-Littlewood and Macdonald polynomials, and define some new families of noncommutative symmetric functions depending on two sequences of parameters.

1. Introduction

There have been several attempts to define analogues of Hall-Litttlewood and Macdonald polynomials in the algebras of noncommutative symmetric functions (**Sym**) and of quasi-symmetric functions ($QSym$). The first analogues of Hall-Littlewood functions were defined by Hivert [4], who replaced symmetrization by a new operation of quasi-symmetrization in Littlewood's original definition. This provided first an interpolation between the monomial and fundamental bases of quasisymmetric functions (which are analogues of the monomial and Schur bases of symmetric functions), by means of a quasi-symmetrizing action of the Hecke algebra. The construction was then further explored on the dual side by combinatorial methods.

It was then shown by Hivert, Lascoux and the third author [5] that these functions admitted a simple and direct combinatorial definition, in which a second parameter could be introduced so as to give analogues of Macdonald polynomials. It was also observed that q and t could be replaced by sequences of indeterminates q_i and t_i in such a way that $q_i = q^i$ and $t_i = t^i$ give back the original version.

Almost simultaneously, similar but different analogues were defined by Bergeron and Zabrocki [1]. Their approach was to obtain Macdonald-like functions from an analogue of the Nabla operator, of which the original Macdonald functions are the eigenvectors.

More recently [10], it has been shown that many more parameters could be introduced in the definition of such bases. Actually, one can have a pair of $n \times n$ matrices (Q_n, T_n) for each degree n. The main properties established in [1] and [5] remain true in this general context, and one recovers the polynomials introduced in these two papers for appropriate specializations of the matrices.

Date: April 25, 2013.

1991 *Mathematics Subject Classification.* 05E05, 16T30.

Key words and phrases. Noncommutative symmetric functions, Quasi-symmetric functions, Macdonald polynomials.

L. Tevlin was partially supported by a grant from the NYU Research Challenge Fund Program.

In the meantime, very different analogues of Hall-Littlewood functions had been defined by Novelli-Thibon-Williams [14]. These analogues, which were based on the monomial functions introduced by the second author [15], had the interesting property that contrary to the other versions, the elements of the (q, t)-Kostka matrices were not monomials, but non-trivial polynomials with an interesting combinatorial interpretation. The approach of [14] was to build q-analogues of product of complete functions (similar to the Q'-version of Hall-Littlewood functions), such that their expansions on some simple q-analogue of the noncommutative fundamental basis of [15] provided combinatorial information on permutation tableaux.

Finally, the Hall-Littlewood basis of [16] was apparently of a different nature. It was defined so as to interpolate between the noncommutative monomial basis introduced in [15] and the ribbon (Schur-like) basis of noncommutative symmetric functions. Introducing certain natural constraints led to a definition involving the special inversion statistic defined in [14]. However, the relation between both approaches was unclear.

The aim of this paper is to clarify the relations between all these different approaches.

We shall uncover the relations between the contructions of [14] and of [16] by first introducing a new multivariate analogue of a classical automorphism of noncommutative symmetric functions, which will allow us to apply the methods of [10] to find a recurrence for the matrices of [16], leading to a natural multiparameter analogue, preserving a factorization property of the Kostka-like matrices.

The present paper started with the observation that the Hall-Littewood basis of [16] could be obtained from one of the bases defined in [14] by applying a version of the so-called $(1 - t)$-transform of noncommutative symmetric functions. A second observation was that the matrices expressing these functions in a suitably modified ribbon basis satisfied a recurrence relation of the same type as those of [10]. Since it has been shown in [10] that such matrices could be defined with many more parameters, we were naturally led to look for a multiparameter version of the functions of [16]. To this aim, we had to find a multiparameter analogue of the $(1-t)$-transform. It turns out that such a map does exist. It is a morphism of algebras, though not of coalgebras, which may be the reason for which it had been overlooked for a long time, despite its simplicity. Its inverse, applied to a complete symmetric function, yields the multiparameter Klyachko element of [9], which, as shown by McNamara and Reutenauer [12] does indeed reduce to a Lie idempotent under an appropriate specialization (see also [2, 13]).

The new recurrence for the (multivariate version of the) Kostka matrices of [16] allows us to provide a proof of a generalization of the product formula annouced in this reference, as well as closed formulas for various other transition matrices. All these results are easily proved by means of the Grassmann algebra formalism of [10]. The techniques of [10] also allow us to introduce a second family of parameters, so as to obtain Macdonald-like bases. For those, we only describe some transition matrices.

Our results do not provide multiparameter analogues of all the constructions of [14], in particular, they do not seem to be related to the combinatorics of permutation

tableaux. So, we conclude with an appendix sketching another approach to introducing more parameters in the constructions of [14], by refining the special inversion statistic with a code in the sense of [6].

Acknowlegements. – Lenny Tevlin would like to thank his co-authors for their warm hospitality at Marne-la-Vallée, where this work was initiated.

2. Notations

Our notations for noncommutative symmetric functions will be as in [3, 9]. Here is a brief reminder.

The Hopf algebra of noncommutative symmetric functions is denoted by **Sym**, or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet. Bases of $\mathbf{Sym}_n$ are labelled by compositions I of n. The noncommutative complete and elementary functions are denoted by S_n and Λ_n, and the notation S^I means $S_{i_1} \dots S_{i_r}$. The ribbon basis is denoted by R_I. The notation $I \vDash n$ means that I is a composition of n. The conjugate composition is denoted by $I^\sim$. The length of I is denoted by $\ell(I)$ and its weight by $|I|$. The mirror image of I is denoted by $\bar{I}$. If $I = (i_1, \dots, i_r)$ is finer than $J = (j_1, \dots, j_s)$, the refining composition I_J is the composition of $r = \ell(I)$ whose kth part is the number of parts of I composing j_k. For example, if $I = (111122311)$ and $J = (3325)$, $I_J = (3213)$.

The product $R_I R_J = R_{I \triangleright J} + R_{IJ}$ of two ribbon Schur functions is the sum of the two terms given by $I \triangleright J = (i_1, \dots, i_r + j_1, \dots, j_s)$ and $IJ = (i_{,1}, \dots, i_r, j_1, \dots, j_s)$.

The graded dual of **Sym** is $QSym$ (quasi-symmetric functions). The dual basis of (S^I) is (M_I) (monomial), and that of (R_I) is (F_I). The *descent set* of $I = (i_1, \dots, i_r)$ is $\mathrm{Des}\,(I) = \{i_1,\ i_1 + i_2, \dots, i_1 + \dots + i_{r-1}\}$.

The monomial basis of [15] is denoted here by Ψ_I (instead of M^I), as in [7]. This basis should not be confused with the basis Ψ^I of [3]. An important convention, followed in all the papers of the NCSF series, is that an upper index denotes a multiplicative basis built on a sequence of generators Z_n: $Z^I = Z_{i_1} \cdots Z_{i_r}$. For a one-part composition $I = (n)$, $Z^{(n)} = Z_n$, and the notation Z_I is used only to denote a non-multiplicative basis such that $Z_{(n)} = Z_n$.

3. Noncommutative Hall-Littlewood functions

This section provides background on some constructions which will be simplified and generalized in the sequel.

3.1. The Hall-Littlewood functions of [16].

The functions $P_I(t)$ of [16] are, for $I = (n)$,

$$P_n(t; A) = \frac{S_n((1-t)A)}{1-t} \tag{1}$$

and for $I = (i_1, \dots, i_r)$, given by the quasideterminant

$$P_I(t;A) = \frac{(-1)^{r-1}}{[r]_t} \begin{vmatrix} P_{i_r} & [1]_t & 0 & \dots & 0 & 0 \\ P_{i_{r-1}+i_r} & P_{i_{r-1}} & [2]_t & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{i_2+\dots+i_r} & \dots & \dots & \dots & P_{i_2} & [r-1]_t \\ \boxed{P_{i_1+\dots+i_r}} & \dots & \dots & \dots & P_{i_1+i_2} & P_{i_1} \end{vmatrix} \tag{2}$$

where $[k]_t = 1 + t + \cdots + t^{k-1}$.

This is not the definition given in [16], but this expression is equivalent to the recurrence

$$[r]_t P_I = P_{i_1} P_{i_2,\dots,i_r} - P_{i_1+i_2} P_{i3,\dots,i_r} + \cdots + (-1)^{r-1} P_{i_1+\cdots+i_r} \,. \tag{3}$$

which is satisfied by the P_I of [16].

The Q-basis is just [16]

$$Q_I = (t;t)_r P_I \,. \tag{4}$$

As already mentioned, the definition of the basis $P_I(t)$ in [16] is equivalent to (1) and (2). This formulation is natural, as it amounts to replacing each Ψ_k in the quasideterminantal definition of the monomial basis Ψ_I by its natural t-analogue, and the integers on the superdiagonal by the corresponding t-integers. Then, by definition, $P_I(1) = \Psi_I$, and it is immediate that $P_I(0) = R_I$, which is the exact analogue of the monomial/Schur specializations of the ordinary Hall-Littlewood P-functions [11].

The main results of [16] are (1) the product rule for $P_I P_J$ (Eq. (86) below), and (2) a combinatorial expression for the analogues of the Kostka polynomials, expressing in the classical case the P-expansion of Schur functions. If we define $K_{IJ}(t)$ by

$$R_J(A) = \sum_I K_{IJ}(t) P_I(t;A) \tag{5}$$

then, the result of [16] is

$$K_{IJ}(t) = \tilde{D}_I^J(t) = t^{\mathrm{maj}(I)} D_I^J(t^{-1}) \tag{6}$$

where $D_I^J(q)$ is the matrix defined in [14, Prop. 3.7] (see the Appendix of the present paper).

3.2. **Comparison with the Hall-Littlewood functions of** [14]**.** In [14], the parameter q plays the role of t^{-1} here. In this reference, a deformation $R_I(q)$ of the ribbon basis is introduced by means of a nonassociative q-product on **Sym**, itself obtained by a linear projection of an associative q-product on the Hopf algebra**WQSym** (based on packed word, or equivalently, set compositions, or surjections, see, *e.g.*, [7, 14]). The $R_J(q)$ are then expanded on a deformation $L_I(q)$ of the fundamental basis L_I (see [15, 7]). The matrix $D_I^J(q)$ is defined by

$$R_J(q) = \sum_I D_I^J(q) \Psi_I(A) \,. \tag{7}$$

Let us define

$$\tilde{R}_J(t) = \sum_I \tilde{D}_I^J(t)\Psi_I(A) \tag{8}$$

$$\tilde{L}_I(t) = \sum_I k_{\bar{I}\bar{J}}(t)\Psi_I(A) \tag{9}$$

$$\tilde{R}_J(t) = \sum_I \tilde{F}_I^J(t)\tilde{L}_I(t) \tag{10}$$

where $k_{IJ}(t)$ is the coefficient of $R_I(A)$ in Hivert's Hall-Littlewood function $H_J(t;A)$, and $F_I^J(t)$ is defined in [14, Eq. (92)].

Then, the relations between the constructions of [14] and [16] can be sumarized as follows:

Proposition 3.1. *Let ϕ_t be the $\mathcal{K}[t]$-linear endomorphism of* **Sym** *defined by $\phi_t(P_I(t)) = \Psi_I$. Then, $\phi_t(R_J) = \tilde{R}_J(t)$.*

It is not clear whether all these families of noncommutative symmetric functions admit compatible multiparameter analogues. A multivariate analogue of $k_{IJ}(t)$ is defined in [5]. But to obtain multivariate versions of all the matrices of [14], we would need a multivariate analogue of $\tilde{R}_I(t)$ compatible with these multivariate k_{IJ} (which is not the case of $\mathcal{R}(\mathbf{t})$ defined below). Numerical experiments indicate that it is unlikely that such an analogue with good compatibility properties with the present multiparameter $\mathcal{P}$-functions could be defined.

However, interesting multivariate analogues of the matrices D_I^J can be defined by a different method, see Section 8.

4. A multivariate analogue of the $(1-t)$-transform

The $(1-t)$-transform is an important automorphism of the algebra of symmetric functions. On power sums, it is defined by $p_n((1-t)X) = (1-t^n)p_n(X)$. This transformation appears for example in the character formula for Hecke algebras, and it is an essential ingredient of the theory of Hall-Littlewood functions [11], where a t-deformed scalar product is defined by

$$\langle f(X), g(X)\rangle_t = \langle f(X), g((1-t)X)\rangle\,. \tag{11}$$

The classical Hall-Littlewood functions come in several flavors. One first defines the functions $P_\lambda(t;X)$. Then, the functions $Q_\lambda(t;X)$ (which are scalar multiples of the P_λ) are defined as their adjoint basis for the t-deformed scalar product. The $Q'_\lambda(t;X)$ are defined as dual to the P_λ for the undeformed scalar product (admitting the Schur functions as an orthonormal basis). The relation between Q and Q' is thus

$$Q_\lambda(t;X) = Q'_\lambda(t;(1-t)X)\,. \tag{12}$$

Most of the known noncommutative analogues of Hall-Littlewood or Macdonald functions admit multiparameter versions [5, 10]. It is known that the $(1-t)$ transform can be defined in **Sym** [9]. If we want to work with multiparameter analogues of the Hall-Littlewood functions, we need a multiparameter version of the $(1-t)$-transform. This can be done as follows.

Recall from [8] the expansion

$$R_I\left((1-t)A)\right) = (-1)^{\ell(I)} \sum_{|J|=|I|, r=\ell(J)} (-1)^r (1-t^{j_r}) t^{\sum_{k\in\mathcal{A}(I,J)} j_k} S^J(A) \tag{13}$$

where

$$\mathcal{A}(I,J) = \{s < \ell(J) | j_1 + \cdots + j_s \notin \mathrm{Des}\,(I)\}. \tag{14}$$

Let $\mathbf{t} = (t_i)_{i\geq 1}$. We introduce the following multivariate version by "lowering the exponents":

$$\mathcal{R}_I(\mathbf{t};A) = (-1)^{\ell(I)} \sum_{|J|=|I|, r=\ell(J)} (-1)^r \left((1-t_{j_r}) \prod_{k\in\mathcal{A}(I,J)} t_{j_k} \right) S^J(A) \tag{15}$$

For example,

$$\mathcal{R}_3 = (1-t_3)S^3 - (1-t_1)t_2 S^{21} - (1-t_2)t_1 S^{12} + (1-t_1)t_1^2 S^{111}\,, \tag{16}$$

$$\mathcal{R}_{21} = -(1-t_3)S^3 + (1-t_1)S^{21} + (1-t_2)t_1 S^{12} - (1-t_1)t_1 S^{111}\,. \tag{17}$$

We can also define

$$\mathcal{S}^I(\mathbf{t};A) = \sum_{J\leq I} \mathcal{R}_J(\mathbf{t};A)\,. \tag{18}$$

Theorem 4.1. *The $\mathcal{S}$-basis is multiplicative:*

$$\mathcal{S}^I(\mathbf{t})\mathcal{S}^J(\mathbf{t}) = \mathcal{S}^{IJ}(\mathbf{t})\,. \tag{19}$$

Thus, $\mathcal{R}_I$ is the image of R_I by the automorphism

$$\theta_{\mathbf{t}}:\ S_n(A) \longmapsto \mathcal{S}_n(\mathbf{t};A)\,. \tag{20}$$

Proof – Let $I \vDash m$ and $J \vDash n$. It is sufficient to prove that $\mathcal{R}_I\mathcal{R}_J = \mathcal{R}_{I\triangleright J} + \mathcal{R}_{IJ}$. Substituting the expressions given by (15) in this product, we get on the one hand

$$\mathcal{R}_I\mathcal{R}_J = \sum_{I',J'} (-1)^{\ell(I)+\ell(J)-r-s} \prod_{k\in\mathcal{A}(I,I') l\in\mathcal{A}(J,J')} t_{i'_k} t_{j'_l} \cdot (1-t_{i'r})(1-t_{j'_s}) S^{I'J'} \tag{21}$$

(where $r = \ell(I')$ and $s = \ell(J')$), and distributing the factor $(1-t_{i'_r})$, this can be rewritten as

$$\begin{aligned} &\sum_K (-1)^{\ell(I)+\ell(J)-r}(1-t_{k_r}) \prod_{p\in\mathcal{A}(IJ,K)} t_{k_p} S^K \\ &\quad + \sum_K (-1)^{\ell(I)+\ell(J)-r-1}(1-t_{k_r}) \prod_{p\in\mathcal{A}(I\triangleright J,K)} t_{k_p} S^K\,, \end{aligned} \tag{22}$$

where K runs over compositions of $m+n$ such that $m \in \mathrm{Des}\,(K)$. On the other hand, we see on (15) that $\mathcal{R}_{I\triangleright J} + \mathcal{R}_{IJ}$ is given by the same expression, where this time K runs over all compositions of $m+n$. But these extra terms cancel, since if $m \notin \mathrm{Des}\,(K)$, then $\mathcal{A}(IJ,K) = \mathcal{A}(I \triangleright J, K)$. ∎

Theorem 4.2. *The inverse of the automorphism* $\theta_{\mathbf{t}} : S_n \mapsto \mathcal{S}_n(\mathbf{t})$ *is*

$$\theta_{\mathbf{t}}^{-1} : S_n \mapsto \mathcal{K}_n(\mathbf{t}; A) = \sum_{I \vDash n} \frac{\prod_{d \in \mathrm{Des}\,(I)} t_d}{(1-t_1)(1-t_2)\cdots(1-t_n)} R_I(A)\,, \tag{23}$$

(the multiparameter Klyachko element already encountered in [9, 12, 2, 13]*).*

Proof – Let $((t))_n := (1-t_1)\cdots(1-t_n)$. Substituting (15) in the expression of $\mathcal{K}_n$, we get

$$\frac{1}{((t))_n} \sum_{I,J \vDash n} (-1)^{\ell(I)+\ell(J)} \prod_{d \in \mathrm{Des}\,(I)} t_d \prod_{k \in \mathcal{A}(I,J)} t_{j_k}(1-t_{j_s}) S^J \tag{24}$$

so that the coefficient of S^J is

$$\frac{(-1)^{s-1}}{((t))_n} \left(\sum_{I \vDash n} \prod_{d \in \mathrm{Des}\,(I)} (-t_d) \prod_{k \in \mathcal{A}(I,J)} t_{j_k} \right) (1-t_{j_s})\,. \tag{25}$$

For $J = (n)$, the sum in the parentheses is $((t))_n$, and for $s = \ell(J) > 1$, the sum vanishes since its terms cancel pairwise as follows. If p is a descent of J, and D is a subset of $[n-1]$ not containing p, there are exactly two compositions I such that $\mathrm{Des}\,(I) \backslash \{p\} = D$, and they have opposite coefficients in the sum. ∎

Note that if we set $t_i = t$ for all i, then $\mathcal{K}_n$ becomes the noncommutative Eulerian polynomial $\mathcal{A}_n^*(t; A)$ of [3].

Example 4.3. The entry (I, J) in the following matrices is the coefficient of S^I in $\mathcal{R}_J(\mathbf{t})$:

$$\begin{pmatrix} 1-t_2 & t_2-1 \\ t_1(t_1-1) & 1-t_1 \end{pmatrix} \qquad \begin{pmatrix} 1-t_3 & t_3-1 & t_3-1 & 1-t_3 \\ t_2(t_1-1) & 1-t_1 & -t_2(t_1-1) & t_1-1 \\ t_1(t_2-1) & -t_1(t_2-1) & 1-t_2 & t_2-1 \\ -t_1^2(t_1-1) & t_1(t_1-1) & t_1(t_1-1) & 1-t_1 \end{pmatrix} \tag{26}$$

The inverse matrices are

$$\begin{pmatrix} \frac{1}{(1-t_1)(1-t_2)} & \frac{1}{(1-t_1)^2} \\ \frac{t_1}{(1-t_1)(1-t_2)} & \frac{1}{(1-t_1)^2} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{(1-t_1)(1-t_2)(1-t_3)} & \frac{1}{(1-t_1)^2(1-t_2)} & \frac{1}{(1-t_1)^2(1-t_2)} & \frac{1}{(1-t_1)^3} \\ \frac{t_2}{(1-t_1)(1-t_2)(1-t_3)} & \frac{1}{(1-t_1)^2(1-t_2)} & \frac{t_1}{(1-t_1)^2(1-t_2)} & \frac{1}{(1-t_1)^3} \\ \frac{t_1}{(1-t_1)(1-t_2)(1-t_3)} & \frac{t_1}{(1-t_1)^2(1-t_2)} & \frac{1}{(1-t_1)^2(1-t_2)} & \frac{1}{(1-t_1)^3} \\ \frac{t_1t_2}{(1-t_1)(1-t_2)(1-t_3)} & \frac{t_1}{(1-t_1)^2(1-t_2)} & \frac{t_1}{(1-t_1)^2(1-t_2)} & \frac{1}{(1-t_1)^3} \end{pmatrix} \tag{27}$$

5. Some sequences of matrices

We shall now introduce some simple sequences of matrices, which will allow us to obtain directly a multiparameter version of the noncommutative Hall-Littlewood functions of [16], as well as their multiparameter Macdonald-like extension.

Let $\mathbf{q} = (q_n)_{n\geq 1}$ and $\mathbf{t} = (t_n)_{n\geq 1}$ be two sequences of commuting indeterminates. For $n \geq 1$, we define three square matrices A_n, B_n, T_n of size 2^{n-1}, indexed by compositions of n arranged in reverse lexicographic order, *e.g.,* for $n = 3$, by $3, 21, 12, 111$ in this order.

Definition 5.1. *For two compositions I, J of n, let $\mathcal{A}(I,J)$ be defined by (14). We define the matrix A_n by*

$$A_n(I,J) = \prod_{k\in\mathcal{A}(\bar I,\bar J)} t_k \times \prod_{l\in\mathcal{A}(\bar I^\sim,\bar J^\sim)} q_l\,. \tag{28}$$

The matrix $T_n = \mathrm{diag}\,(t_{\ell(I)})$ is diagonal. In particular, $T_1 = (t_1)$. We set $A_1 = (1)$, and define B_n as the result of substituting q_{i+1} to q_i in A_n. Thus, $B_1 = (1)$.

Proposition 5.2. *The matrices A_n satisfy the recursion $A_1 = (1)$ and for $n > 1$,*

$$A_n = \begin{pmatrix} B_{n-1} & A_{n-1}T_{n-1} \\ q_1 B_{n-1} & A_{n-1} \end{pmatrix} \tag{29}$$

Proof – Let us cut the matrix A_n into four blocks. Each block is now indexed by pairs of compositions of n which can be represented by compositions of $n-1$.

For example, the top-left corner corresponds to compositions of the form $(1\triangleright I, 1\triangleright J)$. In that case, we have $\mathcal{A}(\overline{1\triangleright I}, \overline{1\triangleright J}) = \mathcal{A}(\bar I, \bar J)$ and $\mathcal{A}(\overline{1\triangleright I}^\sim, \overline{1\triangleright J}^\sim) = \mathcal{A}(\bar I^\sim, \bar J^\sim)+1$ since, for all compositions K, $\overline{1\triangleright K}^\sim = \bar K^\sim \cdot 1$. So the matrix in the top-left corner of A_n is obtained from A_{n-1} by substituting q_{i+1} to q_i.

The same reasoning applies to the other blocks of A_n. ∎

For example,

$$A_2 = \begin{pmatrix} 1 & t_1 \\ q_1 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & t_1 & t_1 & t_1t_2 \\ q_2 & 1 & q_1t_1 & t_2 \\ q_1 & q_1t_1 & 1 & t_1 \\ q_1q_2 & q_1 & q_1 & 1 \end{pmatrix} \tag{30}$$

$$A_4 = \begin{pmatrix} 1 & t_1 & t_1 & t_1t_2 & t_1 & t_1t_2 & t_1t_2 & t_1t_2t_3 \\ q_3 & 1 & q_2t_1 & t_2 & q_2t_1 & t_2 & q_1t_1t_2 & t_2t_3 \\ q_2 & q_2t_1 & 1 & t_1 & q_1t_1 & q_1t_1t_2 & t_2 & t_1t_3 \\ q_2q_3 & q_2 & q_2 & 1 & q_1q_2t_1 & q_1t_2 & q_1t_2 & t_3 \\ q_1 & q_1t_1 & q_1t_1 & q_1t_1t_2 & 1 & t_1 & t_1 & t_1t_2 \\ q_1q_3 & q_1 & q_1q_2t_1 & q_1t_2 & q_2 & 1 & q_1t_1 & t_2 \\ q_1q_2 & q_1q_2t_1 & q_1 & q_1t_1 & q_1 & q_1t_1 & 1 & t_1 \\ q_1q_2q_3 & q_1q_2 & q_1q_2 & q_1 & q_1q_2 & q_1 & q_1 & 1 \end{pmatrix} \tag{31}$$

As for all known analogues of the (q,t)-Kostka matrices, we have:

Proposition 5.3. *The determinants of the A_n are products of linear factors:*

$$\det A_n = \prod_{k=2}^{n}\prod_{i=1}^{n-1}(1-q_it_{k-i})^{\binom{n-1}{k-1}}\,. \tag{32}$$

Proof – Subtracting q_1 times the upper half of the matrix to its lower half, we have

$$\begin{aligned} |A_n| &= \begin{vmatrix} B_{n-1} & A_{n-1}T_{n-1} \\ q_1B_{n-1} & A_{n-1} \end{vmatrix} = \begin{vmatrix} B_{n-1} & A_{n-1}T_{n-1} \\ 0 & A_{n-1} - q_1A_{n-1}T_{n-1} \end{vmatrix} \\ &= |B_{n-1}||A_{n-1}||I_{n-1} - q_1T_{n-1}|\,. \end{aligned} \tag{33}$$

■

For example,

$$|A_4| = (1-q_1t_1)^3(1-q_1t_2)^3(1-q_2t_1)^3(1-q_1t_3)(1-q_2t_2)(1-q_3t_1)\,. \tag{34}$$

6. Macdonald-like polynomials

6.1. **An analogue of the J-basis.** We can now define a basis $\mathcal{J}_K(\mathbf{q},\mathbf{t};A)$ by interpreting the *columns* of A_n as their expansion on the $\mathcal{R}$-basis:

$$\mathcal{J}_K(\mathbf{q},\mathbf{t};A) := \sum_{|I|=n} A_n(I,K)\mathcal{R}_I(\mathbf{t};A)\,. \tag{35}$$

We regard it as an analogue of Macdonald's J-basis.

For example,

$$\begin{aligned}\mathcal{J}_{31}(\mathbf{q},\mathbf{t}) =& t_1\mathcal{R}_4 + \mathcal{R}_{31} + q_2t_1\mathcal{R}_{22} + q_2\mathcal{R}_{211}\\ &+ q_1t_1\mathcal{R}_{13} + q_1\mathcal{R}_{121} + q_1q_2t_1\mathcal{R}_{112} + q_1q_2\mathcal{R}_{1111}.\end{aligned} \tag{36}$$

6.2. **Factorized expressions in the Grassmann algebra.** In [10], a general method for constructing multiparameter bases of noncommutative symmetric functions with a Macdonald-like behaviour has been described. The basic idea is to identify $\mathbf{Sym}_n$ with a Grassmann algebra on $n-1$ variables $\eta_1,\dots,\eta_{n-1}$, a ribbon R_I being encoded by the product $\eta_{d_1}\cdots\eta_{d_k}$, where $\mathrm{Des}\,(I)=\{d_1,\dots,d_k\}$.

We need only a slight modification of the definitions of [10]. Let $U=(u_1,\dots,u_{n-1})$ and $V=(v_1,\dots,v_{n-1})$ be two sequences of parameters. We set

$$\begin{aligned}K_n(U,V) &= (u_1+v_1\eta_1)\cdots(u_{n-1}+v_{n-1}\eta_{n-1})\\ &= \sum_{I\vDash n}\ \prod_{d\in\mathrm{Des}\,(I)} v_d \prod_{e\notin\mathrm{Des}\,(I)} u_e\ R_I\,.\end{aligned} \tag{37}$$

For each composition I of n, build the pair of sequences $(U_I,V_I)=((u_j^I),(v_j^I))_{j=1}^{n-1}$ as follows from the ribbon diagram of I. First, write $(1,q_1),\dots,(1,q_k)$ in this order, starting from the top left cell, in all cells which are non-descents of I. Then, write $(t_1,1),\dots,(t_l,1)$, in this order, in all cells which are descents of I, starting from the bottom right cell, as in the example below.

$$(U_{4121},V_{4121}) = \begin{array}{|c|c|c|c|c} \hline (1,q_1) & (1,q_2) & (1,q_3) & (t_3,1) & \\ \hline \multicolumn{3}{c|}{} & (t_2,1) & \\ \cline{4-5} \multicolumn{3}{c|}{} & (1,q_4) & \multicolumn{1}{c|}{(t_1,1)} \\ \cline{4-5} \multicolumn{4}{c|}{} & \multicolumn{1}{c|}{\times} \\ \cline{5-5} \end{array} \tag{38}$$

Theorem 6.1. *Let $\mathcal{J}'_I(\mathbf{q},\mathbf{t},A)=K_n(U_I,V_I)$. Then,*

$$\mathcal{J}_I(\mathbf{q},\mathbf{t};A) = \theta_{\mathbf{t}}(\mathcal{J}'_I(\mathbf{q},\mathbf{t};A))\,. \tag{39}$$

Proof – By definition, the coefficient of R_I in $\mathcal{J}'_J$ is

$$\prod_{d\in\mathrm{Des}\,(I)} v_d^J \prod_{e\notin\mathrm{Des}\,(I)} u_e^J = \prod_{k\in\mathcal{A}(\bar{I},\bar{J})} t_k \times \prod_{l\in\mathcal{A}(\bar{I}^\sim,\bar{J}^\sim)} q_l = A_n(I,J)\,. \tag{40}$$

∎

As in [10], we also identify $QSym_n$ with a Grassmann algebra on dual variables ξ_i, and set

$$\langle \xi_D, \eta_E\rangle = \delta_{DE}\,. \tag{41}$$

Then, if

$$L_n(X,Y) = (y_1 - x_1\xi_1)\cdots(y_{n-1}-x_{n-1}\xi_{n-1})\,, \tag{42}$$

we have

$$\langle L_n(X,Y), K_n(U,V)\rangle = \prod_{i=0}^{n-1}(u_iy_i - v_ix_i)\,. \tag{43}$$

6.3. **Specialization $\mathbf{q}=0$.** As announced, under the specialization $\mathbf{q}=0$, the $\mathcal{J}$-functions reduce to a multivariate version of the Hall-Littlewood functions of [16], which as we shall see, are simply related to those of [14].

Indeed, Definition 5 of [16] reduces to (13) for $J=(n)$, and expanding the quasi-determinant gives back the general case.

If we regard the $\mathcal{J}$-functions as analogues of the Macdonald J-functions, we can define natural analogues of the classical P and Q-functions by

$$\prod_{i=1}^{\ell(I)}(1-t_i)\mathcal{P}_I(\mathbf{t};A) = \mathcal{Q}_I(\mathbf{t};A) = \mathcal{J}_I(0,\mathbf{t};A) \tag{44}$$

Note that $\mathcal{Q}_n(\mathbf{t};A) = \mathcal{R}_n(\mathbf{t};A)$.

Theorem 6.2. *The $\mathcal{P}$-functions satisfy the recurrence*

$$\frac{1-t_r}{1-t_1}\mathcal{P}_I = \mathcal{P}_{i_1}\mathcal{P}_{i_2,\ldots,i_r} - \mathcal{P}_{i_1+i_2}\mathcal{P}_{i_3,\ldots,i_r} + \cdots + (-1)^{r-1}\mathcal{P}_{i_1+\cdots+i_r}\,. \tag{45}$$

Equivalently, we have the quasideterminantal expression

$$\mathcal{P}_I(\mathbf{t};A) = (-1)^{r-1}\frac{1-t_1}{1-t_r}\begin{vmatrix} \mathcal{P}_{i_r} & 1-t_1 & 0 & \ldots & 0 & 0 \\ \mathcal{P}_{i_{r-1}+i_r} & P_{i_{r-1}} & 1-t_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{P}_{i_2+\ldots+i_r} & \ldots & \ldots & \ldots & \mathcal{P}_{i_2} & 1-t_{r-1} \\ \boxed{\mathcal{P}_{i_1+\ldots+i_r}} & \ldots & \ldots & \ldots & \mathcal{P}_{i_1+i_2} & P_{i_1} \end{vmatrix} \tag{46}$$

Proof – By definition,

$$\mathcal{Q}'_I := \theta_{\mathbf{t}}^{-1}(\mathcal{Q}_I) = K_n(U_I, 0^{n-1})\,. \tag{47}$$

For example,

$$\mathcal{Q}'_{4121} = (t_3+\eta_4)(t_2+\eta_5)(t_1+\eta_7)\,. \tag{48}$$

Thus, the product of a one-part $\mathcal{Q}'_i$ by an arbitrary $\mathcal{Q}'_J$ is

$$\mathcal{Q}'_i\mathcal{Q}'_J = 1\cdot(1+\eta_i)\cdot K_n^{[i]}(U_J, 0^{n-1}) \tag{49}$$

where $K_n^{[i]}$ is obtained from K_n by replacing each η_k by η_{k+i}. Writing

$$1+\eta_i = 1 - t_{\ell(J)} + t_{\ell(J)} + \eta_i\,, \tag{50}$$

we obtain

$$\mathcal{Q}'_i\mathcal{Q}'_J = (1-t_{\ell(J)})\mathcal{Q}'_{i\triangleright J} + \mathcal{Q}'_{IJ}\,. \tag{51}$$

This implies that the $\mathcal{Q}'_I$ and hence also the $\mathcal{Q}_I$ satisfy the recursion

$$\begin{aligned}\mathcal{Q}_I =& \mathcal{Q}_{i_1}\mathcal{Q}_{i_2,\dots,i_r} - (1-t_{r-1})\mathcal{Q}_{i_1+i_2}\mathcal{Q}_{i_3,\dots,i_r}\\ &+ (1-t_{r-2})(1-t_{r-1})\mathcal{Q}_{i_1+i_2+i_3}\mathcal{Q}_{i_4,\dots,i_r} + \cdots + (-1)^{r-1}((t))_{r-1}\mathcal{Q}_{i_1+\cdots+i_r}\,,\end{aligned} \tag{52}$$

which is equivalent to (45). ■

For example,

$$\begin{aligned}\mathcal{Q}'_{4121} =& 1\cdot(1+\eta_4)\cdot(t_2+\eta_7)\\ &- (1-t_3)\cdot 1\cdot(1+\eta_5)\cdot(t_1+\eta_7)\\ &+ (1-t_2)(1-t_3)\cdot 1\cdot(1+\eta_7)\\ &+ (1-t_1)(1-t_2)(1-t_3)\cdot 1\\ &= (t_3+\eta_4)(t_2+\eta_5)(t_1+\eta_7)\,.\end{aligned} \tag{53}$$

Thus, we have:

Corollary 6.3. *For* $\mathbf{t} = (t, t^2, t^3, \dots)$,

$$\mathcal{J}_I(0,\mathbf{t}) = Q_I(t)\,. \tag{54}$$

■

For example,

$$Q_{21}(t) = S^{21}((1-t)A) - (1-t)S^3((1-t)A) = \mathcal{R}_{21}(t) + t\mathcal{R}_3(t)\,. \tag{55}$$

6.4. $\mathcal{J}(\mathbf{q},\mathbf{t})$ **on** S**.** A first remarkable property of the $\mathcal{J}$-basis is its S-expansion. Indeed, as for the Q-functions of [16], the coefficients of the transition matrices are simple products of linear factors and can be explicitly described. For $n = 2, 3$, these are

$$\begin{pmatrix}(1-t_2)(1-q_1) & -(1-t_1)(1-t_2)\\ -(1-t_1)(t_1-q_1) & (1-t_1)(1-t_1^2)\end{pmatrix} \tag{56}$$

(57)

$$\begin{pmatrix}(1-q_1)(1-q_2)(1-t_3) & -(1-q_1)(1-t_1)(1-t_3) & -(1-q_1)(1-t_1)(1-t_3) & (1-t_1)(1-t_2)(1-t_3)\\ -(t_2-q_2)(1-q_1)(1-t_1) & (1-t_1t_2)(1-q_1)(1-t_1) & (t_2-q_1)(1-t_1)^2 & -(1-t_1t_2)(1-t_1)(1-t_2)\\ -(t_1-q_1)(1-q_2)(1-t_2) & (t_1-q_1)(1-t_1)(1-t_2) & (1-t_1^2)(1-q_1)(1-t_2) & -(1-t_1t_2)(1-t_1)(1-t_2)\\ (t_1-q_1)(t_1-q_2)(1-t_1) & -(t_1-q_1)(1-t_1^2)(1-t_1) & -(t_1-q_1)(1-t_1^2)(1-t_1) & (1-t_1t_2)(1-t_1^2)(1-t_1)\end{pmatrix}$$

Equivalently, these matrices describe the expansion of $\mathcal{J}'_J$ on the basis $\theta_{\mathbf{t}}^{-1}(S^I)$. In the Grassmann formalism,

$$\theta_{\mathbf{t}}^{-1}(S^I) = \frac{1}{((t))_I} K_n(1^{n-1}, T_I) \tag{58}$$

where $T_I = (t_j^{(I)})_{j=1}^{n-1}$ is such that $t_j^{(I)} = t_{j-\sum_{k<j} i_k}$ if j is not a descent of I, and $t_j^{(I)} = 1$ otherwise.

Recall the following construction from [10].

We encode a composition I with descent set D by the boolean word $u = (u_1, \dots, u_{n-1})$ such that $u_i = 1$ if $i \in D$ and $u_i = 0$ otherwise.

Let $\mathbf{y} = \{y_u\}$ be a family of indeterminates indexed by all boolean words of length $\leq n-1$. For example, for $n = 3$, we have the six parameters $y_0, y_1, y_{00}, y_{01}, y_{10}, y_{11}$.

Let $u_{m\dots p}$ be the sequence $u_m u_{m+1} \dots u_p$ and define

$$P_I := (1 + y_{u_1}\eta_1)(1 + y_{u_{1\dots 2}}\eta_2) \dots (1 + y_{u_{1\dots n-1}}\eta_{n-1}) \tag{59}$$

or, equivalently,

$$P_I := K_n(1^{n-1} Y_I) \qquad \text{with} \quad Y_I = (y_{u_1}, y_{u_{1\dots 2}}, \dots, y_u) =: (y_k(I)) \,. \tag{60}$$

Similarly, let

$$Q_I := (y_{w_1} - \xi_1)(y_{w_{1\dots 2}} - \xi_2) \dots (y_{w_{1\dots n-1}} - \xi_{n-1}) =: L_n(Y^I, 1^{n-1}) \tag{61}$$

where $w_{1\dots k} = u_1 \dots u_{k-1}\, \overline{u_k}$ where $\overline{u_k} = 1 - u_k$, so that

$$Y^I = (y_{w_1}, y_{w_{1\dots 2}}, \dots, y_{w_{1\dots n-1}}) = (y^k(I)) \,. \tag{62}$$

Then [10, Prop. 4.1], the bases (P_I) and (Q_I) are dual to each other, up to normalization:

$$\langle Q_I, P_J \rangle = \langle L_n(Y^I, 1^{n-1}), K_n(1^{n-1}, Y_J) \rangle = \prod_{k=1}^{n-1} (y^k(I) - y_k(J)) \,, \tag{63}$$

which is indeed zero unless $I = J$.

Thus, in the notation of [10, Eq. (31)]

$$\theta_{\mathbf{t}}^{-1}(S^I) = \frac{1}{((t))_I} P_I, \tag{64}$$

where the parameters y_u of the binary tree are defined by

$$y_u = \begin{cases} 1 & \text{if } u = u'1, \\ t_j & \text{if } u = u'10^j. \end{cases} \tag{65}$$

Since $\mathcal{J}'_J = K_n(U_J, V_J)$, it is sufficient to expand a generic $K_n(U, V)$ on this basis. We have clearly

$$\langle L_n(Y^I, 1^{n-1}), K_n(1^{n-1}, Y_I) \rangle = \delta_{IJ} \prod_{k=1}^{n-1} (y^k(I) - y_k(I)). \tag{66}$$

Thus, the dual basis of $\tilde{P}^I = \theta_{\mathbf{t}}^{-1}(S^I)$ is

$$\tilde{Q}_I = (1 - t_{i_r}) L_n(Y^I, 1^{n-1}), \tag{67}$$

so that the coefficient of S^I in $\mathcal{J}_J$ is

$$(1-t_{i_r})\prod_{k=1}^{n-1}(u_k^J - y^k(I)v_k^J). \tag{68}$$

6.5. $\mathcal{J}$ **on** S **with more parameters.** Although formula (68) is completely explicit, we can better visualize the structure of these matrices by introducing more parameters. Let

$$\mathcal{R}_I(\mathbf{w},\mathbf{x}) := \sum_{J\vDash n}(-1)^{\ell(I)-r}(1-w_r)\prod_{k\in\mathcal{A}(I,J)} x_{j_k} S^J(A). \tag{69}$$

(two sequences of parameters).

If we interpret the above matrices as describing a basis $\mathcal{J}'(\mathbf{q},\mathbf{t})$ on $\mathcal{R}(\mathbf{w},\mathbf{x})$, then $\mathcal{J}'(\mathbf{q},\mathbf{t})$ on S still factorizes in the same manner:

$$\begin{pmatrix} (1-w_2)(1-q_1) & -(1-t_1)(1-w_2) \\ -(1-w_1)(x_1-q_1) & (1-w_1)(1-t_1x_1) \end{pmatrix} \tag{70}$$

(71)

$$\begin{pmatrix} (1-q_1)(1-q_2)(1-w_3) & -(1-q_1)(1-t_1)(1-w_3) & -(1-q_1)(1-t_1)(1-w_3) & (1-t_1)(1-t_2)(1-t_3) \\ -(x_2-q_2)(1-q_1)(1-w_1) & (1-t_1x_2)(1-q_1)(1-w_1) & (x_2-q_1)(1-t_1)(1-w_1) & -(1-t_1x_2)(1-t_2)(1-w_1) \\ -(x_1-q_1)(1-q_2)(1-t_2) & (x_1-q_1)(1-t_1)(1-w_2) & (1-t_1x_1)(1-q_1)(1-w_2) & -(1-t_2x_1)(1-t_1)(1-w_2) \\ (x_1-q_1)(x_1-q_2)(1-w_1) & -(x_1-q_1)(1-t_1x_1)(1-w_1) & -(x_1-q_1)(1-t_1x_1)(1-w_1) & (1-t_1x_2)(1-t_1x_1)(1-w_1) \end{pmatrix}$$

Let us now describe the coefficient g_{JI} of S^J in $\mathcal{J}_I$.

Let $K = I \wedge J$ be the composition whose descent set is $\mathrm{Des}\,(I)\cap\mathrm{Des}\,(J)$. Let $\ell(I)=l$, $\ell(J)=m$ and $\ell(K)=n$. Denote by D and D' the descent sets of K_I and K_J. Define

$$Z_1 = (1-w_{j_m})\prod_{k=1}^{n-1}(1-t_{l-d_k}x_{J_{d'_k}})\prod_{s\in[1,l-1]\backslash D}(1-t_{l-s}). \tag{72}$$

The conjugate mirror compositions $I' = \overline{I}^{\sim}$ and $J' = \overline{J}^{\sim}$ encode the complementary descent sets. Set $K' = I'\wedge J'$, $D'' = \mathrm{Des}\,(K'_{I'})$, and

$$Z_2 = \prod_{k=1}^{\ell(K')-1}(1-q_{d''_k}) \tag{73}$$

Define also $E = [1,m-1]\backslash D'$ and $E' = [1,\ell(I')-1]\backslash D''$. Observe that E and E' have the same cardinality, which we denote by e. Finally, set

$$Z_3 = \prod_{k=1}^{e}(x_{j_{e_k}} - q_{e'_k}). \tag{74}$$

Theorem 6.4. *The coefficient g_{JI} of S^J in $\mathcal{J}_I$ is*

$$g_{JI} = Z = (-1)^{\ell(I)-\ell(J)} Z_1 Z_2 Z_3\,. \tag{75}$$

■

For example, with $I = (13122)$ and $J = (221112)$, we have $K = (4122)$, so that $D = \{2, 3, 4\}$ and $D' = \{2, 3, 5\}$.

Thus, $Z_1 = (1 - w_2)(1 - t_3x_2)(1 - t_2x_1)(1 - t_1x_1)(1 - t_4)$.

Next, $I' = (21321)$, $J' = (1251)$ and $K' = (351)$. Hence, $D'' = \{2, 4\}$ and $Z_2 = (1 - q_2)(1 - q_4)$.

Finally, $E = [1, 4]$ et $E' = [1, 3]$, thus $Z_3 = (x_2 - q_1)(x_1 - q_3)$.

The final result is then

$$Z = -(1-w_2)(1-t_3x_2)(1-t_2x_1)(1-t_1x_1)(1-t_4)(1-q_2)(1-q_4)(x_2-q_1)(x_1-q_3). \tag{76}$$

7. The Hall-Littlewood specialization

Let us now continue the investigation of the specialization $\mathbf{q} = 0$.

7.1. **Transition matrices $\mathcal{S}$ to $\mathcal{Q}$.** The matrices $M(\mathcal{S}, \mathcal{Q})$ have a simple structure For example,

$$\begin{pmatrix} 1 & 1-t_1 \\ . & 1 \end{pmatrix} \tag{77}$$

$$\begin{pmatrix} 1 & 1-t_1 & 1-t_1 & (1-t_1)^2 \\ . & 1 & . & 1-t_2 \\ . & . & 1 & 1-t_1 \\ . & . & . & 1 \end{pmatrix} \tag{78}$$

$$\begin{pmatrix} 1 & 1-t_1 & 1-t_1 & (1-t_1)^2 & 1-t_1 & (1-t_1)^2 & (1-t_1)^2 & (1-t_1)^3 \\ . & 1 & . & 1-t_2 & . & 1-t_2 & . & (1-t_2)^2 \\ . & . & 1 & 1-t_1 & . & . & 1-t_2 & (1-t_1)(1-t_2) \\ . & . & . & 1 & . & . & . & 1-t_3 \\ . & . & . & . & 1 & 1-t_1 & 1-t_1 & (1-t_1)^2 \\ . & . & . & . & . & 1 & . & 1-t_2 \\ . & . & . & . & . & . & 1 & 1-t_1 \\ . & . & . & . & . & . & . & 1 \end{pmatrix} \tag{79}$$

As usual, column J gives $\mathcal{S}^J(\mathbf{t})$ as a linear combination of $\mathcal{Q}_I(\mathbf{t})$. These matrices can also be interpreted as giving the expansion of S^J on the $\mathcal{Q}'_I$.

The S-functions are given by

$$S^J = K_n(1^{n-1}, X_J) \quad \text{where } x_i(J) = \begin{cases} 1 & \text{if } i \in \mathrm{Des}\,(J), \\ 0 & \text{otherwise}. \end{cases} \tag{80}$$

The dual basis of $\mathcal{Q}'_I$ has a simple description:

Theorem 7.1. *In terms of dual Grassmann variables, the dual basis of* $\mathcal{Q}'_I$ *is*

$$\mathrm{G}_I = L_n(A_I, B_I) = (b_1(I) - a_1(I)\xi_1)\cdots(b_{n-1}(I) - a_{n-1}(I)\xi_{n-1}) \tag{81}$$

where

$$(a_j(I), b_j(I)) = \begin{cases} (t_{c(j)}, 1) & \text{if } j \in \mathrm{Des}\,(I), \\ (-1, 0) & \text{otherwise,} \end{cases} \tag{82}$$

where $c(j) = \ell(I) + 1 - r(j)$, $r(j)$ *being the index of the row containing the cell* j *in the ribbon diagram.*

Proof – It is clear that $\langle \mathrm{G}_I, \mathcal{Q}'_I \rangle = 1$, since the only contribution comes from the term of highest degree in $\mathcal{Q}'_I$. It is also clear that $\langle \mathrm{G}_I, \mathcal{Q}'_J \rangle = 0$ if $\mathrm{Des}\,(J) \not\supseteq \mathrm{Des}\,(I)$. In the remaining cases, the greatest descent of J which is not a descent of I will contribute a factor $t_k - t_k$, so that the result is indeed zero. ∎

The coefficient of $\mathcal{Q}_I(\mathbf{t})$ in $\mathcal{S}^J(\mathbf{t})$ is thus

$$\langle \mathrm{G}_I, S^J \rangle = \prod_k (b_k(I)x_k(J) - a_k(I)) \,. \tag{83}$$

For example, the dual basis element corresponding to

$$\mathcal{Q}'_{3122} = (t_3 + \eta_3)(t_2 + \eta_4)(t_1 + \eta_6) \tag{84}$$

is

$$\mathrm{G}_{3122} = (1 - t_4\xi_1)(1 - t_4\xi_2)\xi_3\xi_4(1 - t_2\xi_5)\xi_6(1 - t_1\xi_7) \tag{85}$$

so that the coefficient of $\mathcal{Q}' =_{3122}$ in $S^{11312} = (1 + \eta_1)(1 + \eta_2)(1 + \eta_5)(1 + \eta_6)$ is $(1 - t_4)^2(1 - t_2)$.

7.2. **Multiplicative structure.** An interesting feature of most noncommutative analogues of the P, Q-functions is that their product can be explicitly described. This is again the case here.

Let us recall the product of the $P_I(t)$ is, as stated (along with a few typos) in [15]:

$$\begin{aligned} P_I(t)P_J(t) = \sum_{K \le I} t^{\mathrm{maj}(I_K{}^\sim)} \Bigg(& \begin{bmatrix} \ell(K) + \ell(J) \\ \ell(I) \end{bmatrix}_t P_{K\cdot J}(t) \\ & + \begin{bmatrix} \ell(K) + \ell(J) - 1 \\ \ell(I) \end{bmatrix}_t P_{K \triangleright J}(t) \Bigg) \,, \end{aligned} \tag{86}$$

so that the product of the Q_I is

$$\begin{aligned} Q_I(t)Q_J(t) = \sum_{K \le I} t^{\mathrm{maj}(I_K{}^\sim)} \Bigg(& \frac{[\ell(J)]_t!}{[\ell(J) + \ell(K) - \ell(I)]_t!} Q_{K\cdot J}(t) \\ & + \frac{[\ell(J)]_t!}{[\ell(J) + \ell(K) - \ell(I) - 1]_t!} Q_{K \triangleright J}(t) \Bigg) \,, \end{aligned} \tag{87}$$

with the convention that t-factorials of negative numbers are zero. Note that this amounts to removing the terms Q_C such that $\ell(C) < \ell(I)$.

Now, in the case of multiple t_i, we have a more complicated but still elegant formula.

Let us first define a sequence $S(I,K)$ associated with two compositions, I finer than K. Let $D = \{d_1 < d_2 < \ldots < d_p\}$ be the descent set of $I_K^{\sim}$. Then, $S(I,K) = (d_p, d_{p-1}, \ldots, d_1)$.

For example, with $I = (111122311)$ and $J = (3325)$, $I_K = (3213)$, $I_K^{\sim} = (113211)$ and $D = \{1,2,5,7,8\}$ so that $S(I,K)$ is $[8,7,5,2,1]$. The j-th element of this sequence will be denoted by $s_j^{I,K}$.

Now, thanks to (47), it follows from Theorem 7.1 that

Theorem 7.2.
(88)

$$\mathcal{Q}_I(\mathbf{t})\mathcal{Q}_J(\mathbf{t}) = \sum_{K\le I}\left(\prod_{j=1}^{\ell(I)-\ell(K)} (t_{s_j^{I,K}} - t_{j+s_j^{I,K}+\ell(J)+\ell(K)-\ell(I)})\, \mathcal{Q}_{K\cdot J}(\mathbf{t}) \right.$$
$$\left. +(1-t_{\ell(J)}) \prod_{j=1}^{\ell(I)-\ell(K)} (t_{s_j^{I,K}} - t_{j+s_j^{I,K}+\ell(J)+\ell(K)-\ell(I)-1})\mathcal{Q}_{K\triangleright J}(\mathbf{t})\right)$$

with the convention that terms containing a t_i with $i<0$ are zero. Again, this amounts to removing the terms $\mathcal{Q}_C$ such that $\ell(C)<\ell(I)$.

■

For example,

$$\begin{aligned}\mathcal{Q}_{211}(\mathbf{t})\mathcal{Q}_{21}(\mathbf{t}) &= (t_1-t_3)(t_2-t_3)\mathcal{Q}_{421}(\mathbf{t})\\ &+ (t_2-t_4)\mathcal{Q}_{3121}(\mathbf{t}) + (1-t_2)(t_2-t_3)\mathcal{Q}_{331}(\mathbf{t})\\ &+ (t_1-t_3)\mathcal{Q}_{2221}(\mathbf{t}) + (1-t_2)(t_1-t_2)\mathcal{Q}_{241}(\mathbf{t})\\ &+ \mathcal{Q}_{21121}(\mathbf{t}) + (1-t_2)\mathcal{Q}_{2131}(\mathbf{t}).\end{aligned} \tag{89}$$

$$\begin{aligned}\mathcal{Q}_{111}(\mathbf{t})\mathcal{Q}_{111}(\mathbf{t}) &= (t_1-t_4)(t_2-t_4)\mathcal{Q}_{3111}(\mathbf{t}) + (1-t_3)(t_2-t_3)(t_1-t_3)\mathcal{Q}_{411}(\mathbf{t})\\ &+ (t_2-t_5)\mathcal{Q}_{21111}(\mathbf{t}) + (1-t_3)(t_2-t_4)\mathcal{Q}_{2211}(\mathbf{t})\\ &+ (t_1-t_4)\mathcal{Q}_{12111}(\mathbf{t}) + (1-t_3)(t_1-t_3)\mathcal{Q}_{1311}(\mathbf{t})\\ &+ \mathcal{Q}_{111111}(\mathbf{t}) + (1-t_3)\mathcal{Q}_{11211}(\mathbf{t}).\end{aligned} \tag{90}$$

$$\begin{aligned}\mathcal{Q}_{1211}(\mathbf{t})\mathcal{Q}_{21}(\mathbf{t}) &= (t_2-t_4)(t_3-t_4)\mathcal{Q}_{4121}(\mathbf{t})\\ &+ (1-t_3)(t_3-t_4)\mathcal{Q}_{3221}(\mathbf{t})\\ &+ (t_1-t_3)(t_2-t_3)\mathcal{Q}_{1421}(\mathbf{t})\\ &+ (t_3-t_5)\mathcal{Q}_{31121}(\mathbf{t}) + (1-t_2)(t_3-t_4)\mathcal{Q}_{3131}(\mathbf{t})\\ &+ (t_2-t_4)\mathcal{Q}_{13121}(\mathbf{t}) + (1-t_2)(t_2-t_3)\mathcal{Q}_{1331}(\mathbf{t})\\ &+ (t_1-t_3)\mathcal{Q}_{12221}(\mathbf{t}) + (1-t_2)(t_1-t_2)\mathcal{Q}_{1241}(\mathbf{t})\\ &+ \mathcal{Q}_{111111}(\mathbf{t}) + (1-t_2)\mathcal{Q}_{11211}(\mathbf{t}).\end{aligned} \tag{91}$$

7.3. **Limit cases.** Since $P_I(t)$ of [16] interpolate between ribbon Schur (at $t=0$) and monomial (at $t=1$) bases, it is interesting to investigate these limits in the multivariate version.

Let $\mathbf{b}=(b_n)_{n\geq 1}$ be a sequence of commuting indeterminates. For $I=(i_1,\ldots,i_r)$, define a multivariate deformation of the monomial function Ψ_I by

$$\Psi_I(\mathbf{b};A) = (-1)^{r-1}\frac{b_1}{b_r}\begin{vmatrix} \Psi_{i_r} & 1 & 0 & \ldots & 0 & 0 \\ \Psi_{i_{r-1}+i_r} & \Psi_{i_{r-1}} & \frac{b_2}{b_1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_r} & \ldots & \ldots & \ldots & \Psi_{i_2} & \frac{b_{r-1}}{b_1} \\ \boxed{\Psi_{i_1+\ldots+i_r}} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix} \tag{92}$$

and a deformation of the complete functions by

$$S_r(\mathbf{b};A) = \frac{b_1}{b_r}\begin{vmatrix} \Psi_1 & -\frac{b_{r-1}}{b_1} & 0 & \ldots & 0 & 0 \\ \Psi_2 & \Psi_2 & \frac{b_{r-2}}{b_1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{r-1} & \ldots & \ldots & \ldots & \Psi_1 & -1 \\ \boxed{\Psi_r} & \ldots & \ldots & \ldots & \Psi_2 & \Psi_1 \end{vmatrix}, \tag{93}$$

where Ψ_n is a powers sum (of the first kind) as in [3].

Theorem 7.3. *Consider the specialization $\mathbf{t}=(t^{b_1},t^{b_2},\ldots)$. Then, for $t\to 1$, we have*

$$\mathcal{P}_I(1) = \Psi_I(\mathbf{b}) \quad \textit{and} \tag{94}$$

$$\mathcal{P}_I(0) = R_I(\mathbf{b}), \tag{95}$$

where $R_I(\mathbf{b}) = \sum_{J\preceq I}(-1)^{\ell(I)-\ell(J)}S^J(\mathbf{b})$ as usual.

Proof – This follows from (46). ∎

8. Appendix: another way to introduce multiple parameters in the functions of [14]

The constructions of the present paper do not lead to nice multiparameter analogues of all the matrices defined in [14]. We shall briefly describe here another approach to refining the constructions of [14].

The transition matrices of [14] admit a combinatorial description in terms of a statistic on words called *special inversions*.

This statistic is defined on packed words, *i.e.*, words over the positive integers whose support is an initial interval. Let $u=u_1\cdots u_n$ be a packed word. We say that an inversion $u_i=b>u_j=a$ (where $i<j$ and $a<b$) is *special* if u_j is the *rightmost* occurence of a in u. Let $\mathrm{sinv}(u)$ denote the number of special inversions in u. Note that if u is a permutation, this coincides with its ordinary inversion number.

The *word composition* $\mathrm{WC}(w)$ of a packed word w is the composition whose descent set is given by the positions of the last occurrences of each letter in w. Let $\mathrm{DC}(w)$ be the usual descent composition of w.

For two compositions I and J of the same integer n, let $W(I,J)$ be the set of packed words w such that

$$\mathrm{WC}(w) = I \qquad \text{and} \qquad \mathrm{DC}(w) \leq J. \tag{96}$$

Then, the coefficients $C_I^J(q)$ of [14] are

$$C_I^J(q) = \sum_{w \in W(I,J)} q^{\mathrm{sinv}(w)} \tag{97}$$

and the $D_I^J(q)$ are

$$D_I^J(q) = \sum_{w \in W'(I,J)} q^{\mathrm{sinv}(w)}. \tag{98}$$

where $W'(I,J)$ is the set of packed words w such that

$$\mathrm{WC}(w) = I \qquad \text{and} \qquad \mathrm{DC}(w) = J. \tag{99}$$

The usual inversion number of a permutation can be refined into a list of integers (the Lehmer code) of which it is the sum. Something analogous can be done for special inversions. As in [6], noncommutative generating functions for such codes can be given in the form of flagged ribbon Schur functions.

If $\mathsf{A} = (A_{n_1} \supseteq A_{n_2} \supseteq \ldots \supseteq A_{n_r})$ is a flag of r alphabets, the flagged ribbon Schur function $R_I(\mathsf{A})$, for a composition I of length r, is the sum of all semistandard fillings of the ribbon diagram of I such that only letters of A_{n_i} appear in the ith row.

Let A_i be the ordered alphabet $\{a_0, \ldots, a_i\}$.

Then, (98) (Formula (53) of [14]) can be rewritten as

$$D_I^J(q) = \sum_{w \in W'(I,J)} \left(\prod_{i \in \mathrm{sinv}(w)} a_i \right) \Bigg|_{a_i = q^i}. \tag{100}$$

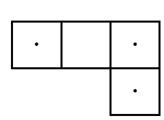
Thus,

$$D_I^J(q) = R_J(\mathsf{A}_{I,J})|_{a_i = q^i} \tag{101}$$

where the flag of alphabets $\mathsf{A}_{I,J}$ is defined in the following way. Let $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$. Then draw the ribbon diagram of I and put dots into the cells corresponding to the descents of J, and also into the last cell of I. Then $\mathsf{A}_{I,J}$ is the sequence of s alphabets $A_{k_1}, \ldots, A_{k_s}$ where k_l is r minus the row number of the l-th dotted cell of I.

For example, with $I = (3,1)$ and $J = (1,2,1)$, the sequence of alphabets is A_1, A_1, A_0 since there are two dotted cells in the first row and one in the second row:

(102)



Here are all the sequences of alphabets (where $(A_{k_1},\dots,A_{k_s})$ has been replaced by $k_1\dots k_s$ to enhance readibility) corresponding to all pairs of compositions of size 3 and 4.

$$\begin{pmatrix} 0 & . & . & . \\ 0 & 10 & 10 & . \\ 0 & . & 10 & . \\ 0 & 10 & 20 & 210 \end{pmatrix} \tag{103}$$

$$\begin{pmatrix} 0 & . & . & . & . & . & . & . \\ 0 & 10 & 10 & . & 10 & 110 & . & . \\ 0 & . & 10 & . & 10 & . & . & . \\ 0 & 10 & 20 & 210 & 20 & 210 & 210 & . \\ 0 & . & . & . & 10 & . & . & . \\ 0 & 10 & 10 & . & 20 & 210 & 210 & . \\ 0 & . & 10 & . & 20 & . & 210 & . \\ 0 & 10 & 20 & 210 & 30 & 310 & 320 & 3210 \end{pmatrix} \tag{104}$$

Note that *all* coefficients of the previous matrix are defined as the flagged ribbons $R_J(\mathsf{A}_I)$, and that some of them vanish: first, the ribbons such that $\ell(J)>\ell(I)$ but also other ribbons, *e.g.*, $R_{21}(A_{12,21}) = R_{21}(A_0,A_0)$ which has to be zero since the word 000 does not have a descent in position 2.

Let now I be a composition and define

$$S^J(\mathsf{A}_{I,J}) := \sum_{J'\leq J} R_{J'}(\mathsf{A}_{I,J'}), \tag{105}$$

then the S^J are multiplicative (as the usual ones).

This implies in particular that the equivalent of the $C_I^J(q)$ (see (29) of [14]) satisfy an induction similar to (50) of [14]: the whole matrix SP_n (see Section 3.5.1 of the same paper) has as analog the matrix of the $S^J(\mathsf{A}_{I,J})$. This explains in a satisfactory way why all its coefficients are products of binomial coefficients. Indeed, if one sends a_i to q^i,

$$S_n(1,q,\dots,q^s) = \begin{bmatrix} s+n \\ n \end{bmatrix}_q . \tag{106}$$

One can also see the picture the other way round:

$$S_n(1,a_1,\dots,a_s) \tag{107}$$

is the natural multivariate analog of the binomial $\binom{n+s}{n} = S^n(s+1)$ (in λ-ring notation). Indeed, the induction on the S_n is the natural analog of the classical induction on binomials:

$$S_n(1,a_1,\dots,a_s) = S_n(1,a_1,\dots,a_{s-1}) + S_{n-1}(1,a_1,\dots,a_s)a_s. \tag{108}$$

Setting $a_i = q_i$ (commuting parameters), we obtain multiparameter versions of the Hall-Littlewood functions of [14]. Their Macdonald-like extensions are not known.

References

[1] N. Bergeron and M. Zabrocki, *q and q,t-analogs of non-commutative symmetric functions*, Discrete Math. **298** (2005), no. 1-3, 79–103.

[2] A. Dzhumadil'daev, *Lie expression for multi-parameter Klyachko idempotent*, J. Algebraic Combin. **33** (2011), 531–542.

[3] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), 218–348.

[4] F. Hivert, *Hecke algebras, difference operators, and quasi-symmetric functions*, Adv. Math. **155** (2000), 181–238.

[5] F. Hivert, A. Lascoux and J.-Y. Thibon, *Noncommutative symmetric functions with two and more parameters*, preprint arXiv: math.CO/0106191.

[6] F. Hivert, J.-C. Novelli, and J.-Y. Thibon,*Multivariate generalizations of the Foata-Schutzenberger equidistribition* , Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities, DMTCS Proceedings, 2006 (electronic).

[7] F. Hivert, J.-C. Novelli, L. Tevlin, and J.-Y. Thibon, *Permutation statistics related to a class of noncommutative symmetric functions and generalizations of the Genocchi numbers*, Selecta Math. (N.S.) 15 (2009), no. 1, 105–119.

[8] M. Josuat-Vergès, J.-C. Novelli and J.-Y. Thibon, *The algebraic combinatorics of snakes*, arXiv:1110.5272v1.

[9] D. Krob, B. Leclerc and J.-Y. Thibon, *Noncommutative symmetric functions II: Transformations of alphabets*, Internat. J. Alg. Comp. **7** (1997), 181–264.

[10] A. Lascoux, J.-C. Novelli and J.-Y. Thibon, *Noncommutative symmetric functions with matrix parameters*, arXiv:1110.3209.

[11] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford, 1995.

[12] P. McNamara and C. Reutenauer, *P-Partitions and a Multi-Parameter Klyachko Idempotent*, Electronic J. Combin. **11**(2) (2005), #R21 (18 pp.).

[13] J.-C. Novelli and J.-Y. Thibon, *Free quasi-symmetric functions and descent algebras for wreath products, and noncommutative multisymmetric functions*, Discrete Math. **310** (2010), 3584-30606.

[14] J.-C. Novelli, J.-Y. Thibon and L.K. Williams, *Combinatorial Hopf algebras, noncommutative Hall-Littlewood functions, and permutation tableaux*, Advances in Math. **224** (2010), 1311-1348.

[15] L. Tevlin, *Noncommutative Monomial Symmetric Functions*, Proc. FPSAC'07, Tianjin, China.

[16] L. Tevlin, *Noncommutative Symmetric Hall-Littlewood Polynomials*, Proc. FPSAC 2011, DMTCS Proc. **AO** 2011, 915–926.

(J.-C Novelli, J.-Y. Thibon) Université Paris-Est Marne-la-Vallée, Laboratoire d'Informatique Gaspard-Monge, 5 Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France

(L. Tevlin) Liberal Studies,, New York University, 726 Broadway, New York, N.Y. 10003, U.S.A.